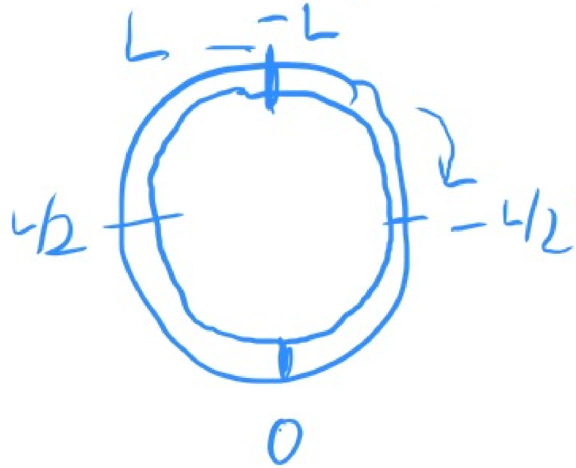
Last time: heat equation for ring



circumference $2L$

identify ring with interval $[-L, L]$

got boundary conditions

$$u(-L,t) = u(L,t)$$

$$\frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t)$$

come from fact that $-L$ and L indicate same point.

$$u(x,t) = \phi(x)\, G(t)$$

$$(BC) \Rightarrow \quad \phi(-L) = \phi(L)$$

$$\phi'(-L) = \phi'(L)$$

$$\phi''(x) = -\lambda\, \phi(x)$$

separation of variables $\Rightarrow$ ODE's

$$G'(t) = -\lambda k\, G(t)$$

general sol. for ϕ:

$$\phi(x) = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x$$

possible eigenvalues λ ?

look at boundary condition:

$$c_1 \cos\sqrt{\lambda}(-L) + c_2 \sin\sqrt{\lambda}(-L) = \phi(-L) = \phi(L) = \cancel{c_1 \cos\sqrt{\lambda} L} + c_2 \sin\sqrt{\lambda} L$$

'' cos even '' sin odd ''

$$\cancel{c_1 \cos\sqrt{\lambda} L} - c_2 \sin\sqrt{\lambda} L \quad = \quad ''$$

$$\Rightarrow \boxed{2 c_2 \sin\sqrt{\lambda} L = 0}$$

as before: try find coefficients a_n and b_n

such that

$$a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

$= f(x) =$ function of initial condition.

$u(x,0) = f(x)$

$$\boxed{\begin{aligned} a_0 &= \frac{1}{2L} \int_{-L}^{L} f(x)\,dx \\ a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x\,dx \\ b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x\,dx \end{aligned}}$$

Solution of IC & BC & PDE

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x\, e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

$$+ \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x\, e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

aside (Section 2.2.)

useful point of view:

consider taking (partial) derivatives as a map from functions to functions

these are called <u>differential operators</u>

e.g. $u(x,t) \longmapsto \frac{\partial u}{\partial x}(x,t)$

or $u(x,t) \longmapsto \frac{\partial u}{\partial t}(x,t)$

<u>Def.</u> An operator on functions is called <u>linear</u> if $L(c_1 u_1 + c_2 u_2) = c_1 L(u_1) + c_2 L(u_2)$

c_1, c_2 constants, u_1, u_2 functions

Examples: (1) Any differential operator is linear

e.g. $\partial/\partial x$: $u \to \dfrac{\partial u}{\partial x}$

$\partial^2/\partial x^2$: $u \to \partial^2 u/\partial x^2$

(2) also operators of form $a(x,t)\,\partial/\partial x$:

$u(x,t) \mapsto a(x,t)\,\partial u/\partial x\,(x,t)$

are linear

Def. Let L be a linear operator

Equations of the form $L(u)=0$

are called **homogeneous** equations

Theorem Let L be a linear operator

If u_1 and u_2 are solutions of hom. equ. $L(u)=0$

$\Rightarrow L(c_1 u_1 + c_2 u_2)=0$ i.e. $c_1 u_1 + c_2 u_2$ is a solution

Ex. $L(u) = \dfrac{\partial u}{\partial t} - k \cdot \dfrac{\partial^2 u}{\partial x^2}$

$\Rightarrow L(u)=0 \Leftrightarrow \dfrac{\partial u}{\partial t} - k \dfrac{\partial^2 u}{\partial x^2} = 0$

$\Leftrightarrow \partial u/\partial t = k\, \partial^2 u/\partial x^2$ i.e. u sol. heat equation